# Track Fitting on the Riemann Sphere

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**Abstract**

We present a novel method of fitting trajectories of charged particles in high-energy physics particle detectors. The method fits a circular arc to two-dimensional measurements by mapping the measurements onto the Riemann sphere and fitting a plane to the transformed coordinates of the measurements. In this way, the non-linear task of circle fitting, which in general requires the application of some iterative procedure, is turned into a linear problem which can be solved in a fast, direct and non-iterative manner.

We illustrate the usefulness of our approach by stating results from a simulation experiment of tracks from the ATLAS Inner Detector Transition Radiation Tracker (TRT). The experiment shows that the accuracy of the estimated track parameters is virtually as good as the accuracy obtained by applying an optimal, non-linear least-squares procedure. The computational cost of our method, however, is significantly lower.

Keywords: Track fitting, Riemann sphere

## 1 Introduction

In high-energy physics experiments, the task of fitting measurements to circular arcs is of vital importance. This is due to the fact that particle tracks in homogeneous magnetic fields are helices, or in the bending plane, circles. The particle momenta are directly connected to the curvature of these circles, so the circle parameters have to be estimated as accurately as possible.

If the measurement errors are Gaussian, the least-squares method is the optimal one. For circles, however, it is in general not possible to express the measurements as linear functions of the circle parameters. The standard tools of linear regression can therefore not be used directly, and one has to rely on non-linear approaches as for instance the iterative Gauss-Newton method [1]. Due to the iterative nature of this class of methods, the computational cost might be quite high. In addition, one has to supply the algorithm with an initial guess of the parameters of the circle.

In this work, we present a novel method of track fitting based on mapping two-dimensional measurements onto the Riemann sphere [2]. It is known from complex analysis that circles and lines in the plane uniquely map onto circles on the Riemann sphere [3]. Since a circle on the Riemann sphere unambiguously defines a plane in space, there is a one-to-one correspondence between circles and lines in the plane and planes in space. The task of fitting points to a circular arc is therefore mapped onto the problem of — in a least-squares sense — fitting the transformed points to a plane in space. This can be done in a fast and non-iterative manner, and there is no need for any initialization of the track parameters.

## 2 Basic concepts

A point $(R_i, \phi_i)$ in the plane is mapped onto the point $(x_i, y_i, z_i)$ on the Riemann sphere by

$$
\begin{aligned}
x_i &= R_i \cos\phi_i/(1+R_i^2), && (1)\\
y_i &= R_i \sin\phi_i/(1+R_i^2), && (2)\\
z_i &= R_i^2/(1+R_i^2). && (3)
\end{aligned}
$$

A plane in space is described by a set of four parameters $\{c, n_1, n_2, n_3\}$, where $\boldsymbol{n}^T = (n_1, n_2, n_3)$ is a unit length normal vector of the plane and $c$ is a signed distance from the plane to the origin. Fitting a plane in space to the number $N$ measurements on the Riemann sphere will be defined as the minimum of

$$S = \sum_{i=1}^{N} (c + n_1 x_i + n_2 y_i + n_3 z_i)^2 = \sum_{i=1}^{N} d_i^2 \tag{4}$$

with respect to $c$, $n_1$, $n_2$ and $n_3$, subject to the constraint $n_1^2 + n_2^2 + n_3^2 = 1$. Thus, we want to minimize the sum of squared distances $d_i^2$ from the points to the plane. In order to tackle this problem, we first solve $\partial S/\partial c = 0$. This gives

$$c = -\boldsymbol{n}^T \overline{\boldsymbol{r}}, \tag{5}$$

with $\overline{\boldsymbol{r}}^T = (\overline{x}, \overline{y}, \overline{z})$. Here $\overline{\boldsymbol{r}}$ can be interpreted as the mean vector of the data, with $\overline{x} = \sum_i x_i/N$, $\overline{y} = \sum_i y_i/N$ and $\overline{z} = \sum_i z_i/N$. Omitting an unimportant constant factor $N$, the cost function $S$ can now be written

$$S = \boldsymbol{n}^T \boldsymbol{A} \boldsymbol{n}, \tag{6}$$

where

$$\boldsymbol{A} = \frac{1}{N} \cdot \sum_{i=1}^{N} (\boldsymbol{r}_i - \overline{\boldsymbol{r}})(\boldsymbol{r}_i - \overline{\boldsymbol{r}})^T, \tag{7}$$

and $\boldsymbol{r}_i^T = (x_i, y_i, z_i)$. The matrix $\boldsymbol{A}$ can be recognized as the sample covariance matrix of the measurements. The vector $\boldsymbol{n}$ minimizing $S$ is the eigenvector of $\boldsymbol{A}$ corresponding to its smallest eigenvalue. After having determined the normal vector $\boldsymbol{n}$, $c$ can be directly found from Equation (5).

The mapping from the parameters of the plane to one set of circle parameters — the centre coordinates $(u_0, v_0)$ and the radius of curvature $\rho$ — is given by

$$
\begin{aligned}
u_0 &= -\frac{n_1}{2\,(c+n_3)}, && (8)\\
v_0 &= -\frac{n_2}{2\,(c+n_3)}, && (9)\\
\rho^2 &= \frac{n_1^2 + n_2^2 - 4c\,(c+n_3)}{4\,(c+n_3)^2}. && (10)
\end{aligned}
$$

There is a singularity for $c = -n_3$, something which arises in the straight line limit. In this case it is mandatory to map the parameters of the plane to another set of track parameters which is well-defined also for straight lines. This can always be done.

Using the transformation formulas in Equations (1), (2) and (3), it is possible to relate the distances from the measurements in the plane to the circle to the distances from the transformed measurements to the corresponding plane in space. It turns out that the orthogonal or shortest distance $d_{i,\perp}$ from the point $i$ to the circle is proportional to $\left(1 + R_i^2\right) \cdot d_i$, where $d_i$ is the distance from the transformed measurement to the plane. In the simulation experiment of this paper, the magnitude of the measurement error is constant throughout the detector. Thus, the following, modified cost function,

$$S_{\text{mod}} = \sum_{i=1}^{N} \left(1 + R_i^2\right)^2 \cdot d_i^2, \tag{11}$$

should to a good approximation fulfil the requirements of the Gauss-Markov conditions. The minimum of $S_{\text{mod}}$ with respect to the plane parameters can be found using exactly the same techniques as the ones described above.

## 3 A simulation experiment in the ATLAS TRT

ATLAS is one of the two general-purpose experiments in the Large Hadron Collider (LHC) project at CERN. The TRT is one of the sub-systems of the ATLAS Inner Detector. Herein, we have focused on tracks going through the barrel part of the TRT. In the barrel, the drift tubes or "straws" are arranged in cylindrical layers, each straw being parallel to the beam axis. There are $75$ straw layers and about $50000$ drift tubes in total in the TRT [4]. Since the basic detector elements are drift tubes, the measurements are in principle ambiguous. In our experiment, however, we have mainly been interested in evaluating basic statistical properties of different types of estimators. The mirror hits have therefore been turned off during the simulation.

Our track sample consists of $9800$ tracks coming from the origin, all with a transverse momentum $p_T$ larger than $1$ GeV/c. All material effects have been neglected, but a measurement error of $250$ $\mu$m has been simulated. Some of the methods tested out need an initial estimate of the track parameters. This has been provided by a least-squares fit to a straight line in the $R\phi$-projection. The precision of the methods is evaluated by the generalized variance $V$ of the residuals of the estimated track parameters with respect to the true values.

The results of the experiment are shown in Table I. We here state the relative generalized variance $V_{\text{rel}}$ and the CPU time consumption $t_{\text{rel}}$ relative to the method with the best performance. We have focused on three methods: an iterative, non-linear least-squares procedure (NLS) based on the Levenberg-Marquardt algorithm [1], a Kalman filter (KF) [5] and the least-squares fit on the Riemann sphere (RF). For the first two algorithms, the CPU time consumption is given both with and without the track parameter initialization procedure. For the Riemann fit, the generalized variance is given for the two cases with and without the modifications mentioned in Section 2.

| Method | $V_{\text{rel}}$ | $t_{\text{rel}}$ |
|---|---|---|
| NLS without initialization | 1.000 | 36.3 |
| NLS with initialization | 1.000 | 41.4 |
| KF without initialization | 1.001 | 28.2 |
| KF with initialization | 1.001 | 33.3 |
| RF with modifications | 1.003 | 1.00 |
| RF without modifications | 1.055 | 1.00 |

**Table I:** The relative generalized variance and the relative time consumption for three different algorithms.

As expected, the NLS method is the most accurate. However, the Kalman filter and the RF with modifications are negligibly less accurate. The KF estimates are obtained in an approximation of linear track parameter extrapolation. The original RF is also performing well, but the difference to the first two methods is significant. With the modifications included, it is virtually as good as the NLS and the KF.

Concerning CPU time consumption, the RF is by far the fastest algorithm. This is true also if we neglect the track parameter initialization. It has to be noted that this initialization is a vital part of the two other methods. In order to make a realistic comparison, the initialization should therefore be included. Due to some overhead in the code, this procedure is slower than what is expected. Since the initialization basically is a linear least-squares fit, it should be possible to make it as fast as the RF. Being an iterative method, the NLS is the slowest of all. The KF is also significantly slower than the RF. If process noise cannot be neglected, however, the KF is able to include this in a natural and efficient way and is therefore the natural choice.

## 4 Conclusions and outlook

We have in this paper presented a novel algorithm for track fitting in high-energy particle physics detectors, based on the idea of mapping two-dimensional measurements onto the Riemann sphere. We have also introduced some modifications of the ideas from the original paper [2]. This results in a method which is virtually as precise and much faster than an optimal, non-linear procedure, and it is also significantly faster than the Kalman filter.

If the mirror hits are turned on, the fit has to be supplemented by a decision procedure which selects the correct hits and rejects the mirror hits. We have developed an adaptive version of the Riemann fit which is called the Elastic Planes algorithm [6]. It has been shown in extensive simulation studies that it has the same precision as the Elastic Arms algorithm [7] and the Deterministic Annealing Filter [8] if multiple scattering is negligible, and that it is much faster than either of those.

When dealing with three-dimensional measurements, one will rather fit a helix than a circle to the data. It should be possible to extend the Riemann fit to this more general situation without losing the advantages of a fast and direct solution method. This will be the topic of a future study.

## References


1 A. Sen and M. Srivastava, *Regression Analysis*. Springer-Verlag, New York, 1990.

2 B. Lillekjendlie, Circular arcs fitted on a Riemann sphere. *Computer Vision and Image Understanding* 67 (1997) 311.

3 L. V. Ahlfors, *Complex analysis: an introduction to the theory of analytic functions of one complex variable*. Int. series in pure and applied mathematics, McGraw-Hill, 1979.

4 *ATLAS Inner Detector Technical Design Report*. CERN/LHCC 97-16, CERN, Geneva, 1997.

5 R. Frühwirth, Application of Kalman filtering to track and vertex fitting. *Nuclear Instruments and Methods* A262 (1987) 444.

6 A. Strandlie, J. Wroldsen, R. Frühwirth and B. Lillekjendlie, Particle tracks fitted on the Riemann sphere. Submitted to Computer Physics Communications.

7 M. Ohlsson, C. Peterson and A. Yuille, Track finding with deformable templates — the elastic arms approach. *Computer Physics Communications* 71 (1992) 77.

8 R. Frühwirth and A. Strandlie, Track fitting with ambiguities and noise: a study of elastic tracking and non-linear filters. *Computer Physics Communications* 120 (1999) 197.